RETRACTS OF SIGMA-PRODUCTS OF HILBERT CUBES

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Abstract. We consider the sigma-product of the ω_1-power of the Hilbert cube. This space is characterized among its retracts as the only one without G_δ-points.

1. Introduction

Sigma-products and their subspaces have been extensively studied by topologists and functional analysts for several decades. We refer the reader to [4] where a comprehensive survey of the related results from both topology and functional analysis are discussed in detail.

Recall that the sigma-product $\Sigma(X,*)$ of an uncountable collection $\{X_t \colon t \in T\}$ of spaces with base points $*_t \in X_t$, $t \in T$, is the subspace of the product $X = \prod\{X_t \colon t \in T\}$ defined as follows

$$\Sigma(X,*) = \{\{x_t \colon t \in T\} \in X \colon |\{t \in T \colon x_t \neq *_t| \leq \omega\}\}.$$

We are interested in the case when each X_t is a copy of the Hilbert cube I^ω, $|T| = \omega_1$ and $*_t$ is the point (in I^ω) all coordinates of which equal to 0. The corresponding sigma-product is denoted by Σ.

Our main result (Theorem 3.1) states that if a retract of Σ does not contain G_δ-points, then it is homeomorphic to Σ.

2. Auxiliary lemmas

Terminology, notation and results related to inverse spectra and absolute retracts used here can be found in [1]. One of the main concepts we need below is that of ω-spectra $\mathcal{S} = \{X_\alpha, p_\alpha^\beta, A\}$. These are ω-continuous inverse spectra consisting of metrizable compact spaces X_α, surjective projections $p_\alpha^\beta \colon X_\beta \to X_\alpha$, $\alpha \leq \beta$, and an ω-complete indexing set A. This essentially means that A contains supremums of countable chains and that for any such chain $\{\alpha_n \colon n \in \omega\}$ the space X_α, where $\alpha = \sup\{\alpha_n \colon n \in \omega\}$, is naturally homeomorphic to the limit of the inverse sequence $\mathcal{S}_\alpha = \{X_{\alpha_n}, p_{\alpha_n}^{\alpha_{n+1}}, \omega\}$.

Recall also that a compact space is an absolute retract if and only if it is a retract of a Tychonov cube and that a map $p \colon X \to Y$ of compact spaces is soft

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if for any compactum B, it's closed subset A and maps g and h such that the following diagram (of undotted arrows) commutes

$$\begin{tikzcd} X \arrow[r, "p"] & Y \\ A \arrow[u, "g"] \arrow[r, hook, "i"] & B \arrow[u, "h"'] \arrow[ul, dotted, "k"'] \end{tikzcd}$$

there exists a map $k\colon B \to X$ (the dotted arrow) such that $k|A = g$ and $fk = h$.

The prime example of a soft map is the projection $X \times I^{\omega} \to X$.

Finally recall that for a given map $p\colon X \to Y$ a closed subset $F \subseteq X$ is a fibered Z-set in X (with respect to p) if for any open cover $\mathcal{U} \in \operatorname{cov}(X)$ there exists a map $f_{\mathcal{U}}\colon X \to X$ such that $pf_{\mathcal{U}} = p$ (i.e. $f_{\mathcal{U}}$ acts fiberwise), $f_{\mathcal{U}}(X) \cap F = \emptyset$ and $f_{\mathcal{U}}$ is $\mathcal{U}$-close to id_X.

Lemma 2.1. *Let a non-metrizable compact space X be represented as the limit of an ω-spectrum $\mathcal{S} = \{X_{\alpha}, p_{\alpha}^{\beta}, A\}$ with soft projections p_{α}^{β}. Suppose that F is a closed subset of X containing no closed G_{δ}-subsets of X. Then for each $\alpha \in A$ there exists $\beta \in A$, with $\beta > \alpha$, such that there is a map $i_{\alpha}^{\beta}\colon X_{\alpha} \to X_{\beta}$ satisfying the following two properties:*

(1) $p_{\alpha}^{\beta} i_{\alpha}^{\beta} = \mathrm{id}_{X_{\alpha}}$,
(2) $i_{\alpha}^{\beta}(X_{\alpha}) \cap p_{\beta}(F) = \emptyset$.

Proof. Let $\alpha_0 = \alpha$ and $x_0 \in X_{\alpha_0}$. Since $p_{\alpha_0}^{-1}(x_0)$ is closed and G_{δ} in X, it follows from the assumption that $p_{\alpha_0}^{-1}(x_0) \setminus F \neq \emptyset$. Take an index $\alpha_1 \in A$ such that $\alpha_1 > \alpha_0$ and $(p_{\alpha_0}^{\alpha_1})^{-1}(x_0) \setminus p_{\alpha_1}(F) \neq \emptyset$. Let $x_1 \in (p_{\alpha_0}^{\alpha_1})^{-1}(x_0) \setminus p_{\alpha_1}(F)$. The softness of the projection $p_{\alpha_0}^{\alpha_1}\colon X_{\alpha_1} \to X_{\alpha_0}$ guarantees the existence of a map $i_0^1\colon X_{\alpha_0} \to X_{\alpha_1}$ such that $p_{\alpha_0}^{\alpha_1} i_0^1 = \mathrm{id}_{X_{\alpha_0}}$ and $i_0^1(x_0) = x_1$. Let

$$V_1 = \{x \in X_{\alpha_0}\colon i_0^1(x) \notin p_{\alpha_1}(F)\}.$$

Note that $x_0 \in V_1$ and consequently V_1 is a non-empty open subset of X_{α_0}.

Let $\gamma < \omega_1$. Suppose that for each λ, $1 \leq \lambda < \gamma$, we have already constucted an index $\alpha_{\lambda} \in A$, an open subset $V_{\lambda} \subseteq X_{\alpha_0}$ and a section $i_0^{\lambda}\colon X_{\alpha_0} \to X_{\alpha_{\lambda}}$ of the projection $p_{\alpha_0}^{\alpha_{\lambda}}\colon X_{\alpha_{\lambda}} \to X_{\alpha_0}$, satisfying the following conditions:

(i) $\alpha_{\lambda} < \alpha_{\mu}$, wherenever $\lambda < \mu < \gamma$,
(ii) $\alpha_{\mu} = \sup\{\alpha_{\lambda}\colon \lambda < \mu\}$, whenever $\mu < \gamma$ is a limit ordinal,
(iii) $V_{\lambda} \subsetneqq V_{\mu}$, whenever $\lambda < \mu < \gamma$,
(iv) $V_{\mu} = \cup\{V_{\lambda}\colon \lambda < \mu\}$, whenever $\mu < \gamma$ is a limit ordinal,
(v) $i_0^{\mu} = \triangle\{i_0^{\lambda}\colon \lambda < \mu\}$, whenever $\mu < \gamma$ is a limit ordinal,
(vi) $i_0^{\lambda} = p_{\alpha_{\lambda}}^{\alpha_{\mu}} i_0^{\mu}$, whenever $\lambda < \mu < \gamma$,
(vii) $V_{\lambda} = \{x \in X_{\alpha_0}\colon i_0^{\lambda}(x) \notin p_{\alpha_{\lambda}}(F)\}$.

We shall construct the index α_{γ}, the open subset $V_{\gamma} \subseteq X_{\alpha_0}$ and the section $i_0^{\gamma}\colon X_{\alpha_0} \to X_{\alpha_{\gamma}}$ of the projection $p_{\alpha_0}^{\alpha_{\gamma}}\colon X_{\alpha_{\gamma}} \to X_{\alpha_0}$.

Suppose that γ is a limit ordinal. By (i), $\{\alpha_\mu\colon \mu < \gamma\}$ is a countable chain in A and we let (recall that the indexing set A of $\mathcal{S}$ is a ω-compelete set and therefore contains supremums of countable chains of its elements)

$$\alpha_\gamma = \sup\{\alpha_\mu\colon \mu < \gamma\} \in A.$$

By the ω-continuity of the spectrum $\mathcal{S}$, the compactum X_{α_γ} is naturally homeomorphic to the limit of the inverse sequence $\{X_{\alpha_\mu}, p^{\alpha_\mu}_{\alpha_\lambda}, \lambda, \mu < \gamma\}$. Consequently, by (vi), the diagonal product

$$i^\gamma_0 = \triangle\{i^\mu_0\colon \mu < \gamma\}\colon X_{\alpha_0} \to X_{\alpha_\gamma}$$

is well-defined and satifies corresponding conditions (v) and (vi). Let

$$V_\gamma = \{x \in X_{\alpha_0}\colon i^\gamma_0(x) \notin p_{\alpha_\gamma}(F)\}.$$

Note that $V_\gamma = \cup\{V_{\alpha_\mu}\colon \mu < \gamma\}$. Then, corresponding conditions (vii), (iii) and (iv) are satisfied.

Next consider the case $\gamma = \mu + 1$. In case $V_\mu = X_{\alpha_0}$, the desired β is α_μ. Suppose that $V_\mu \neq X_{\alpha_0}$ and let

$$x_\mu = i^\mu_0(z) \in i^\mu_0(X_{\alpha_0}) \subseteq X_{\alpha_\mu},$$

where $z \in X_{\alpha_0} \setminus V_\mu$. Since $p^{-1}_{\alpha_\mu}(x_\mu)$ is closed and G_δ in X, we have $p^{-1}_{\alpha_\mu}(x_\mu) \setminus F \neq \emptyset$ (note that X_{α_μ} is a metrizable compactum). Choose an index $\alpha_\gamma \in A$ so that $\alpha_\gamma > \alpha_\mu$ and $(p^{\alpha_\gamma}_{\alpha_\mu})^{-1}(x_\mu) \setminus p_{\alpha_\gamma}(F) \neq \emptyset$.

Softness of the projection $p^{\alpha_\gamma}_{\alpha_\mu}\colon X_{\alpha_\gamma} \to X_{\alpha_\mu}$ guarantees the existence of a map $i^\gamma_\mu\colon X_{\alpha_\mu} \to X_{\alpha_\gamma}$ such that $p^{\alpha_\gamma}_{\alpha_\mu} i^\gamma_\mu = \mathrm{id}_{X_{\alpha_\mu}}$ and $i_\mu(x_\mu) = z'$, where $z' \in (p^{\alpha_\gamma}_{\alpha_\mu})^{-1}(x_\mu) \setminus p_{\alpha_\gamma}(F)$. Let $i^\gamma_0 = i^\gamma_\mu i^\mu_0$ and $V_\gamma = \{x \in X_{\alpha_0}\colon i^\gamma_0(x) \notin p_{\alpha_\gamma}(F)\}$. Note that $V_\mu \subseteq V_\gamma$ and $z' \in V_\gamma \setminus V_\mu$. This completes construction of the needed objects in the case $\gamma = \mu + 1$.

Thus the construction can be carried out for each $\lambda < \omega_1$ and we obtain a strictrly increasing collection $\{V_\lambda\colon \lambda < \omega_1\}$ of open subsets of the metrizable compactum X_{α_0}. Clearly, this collection must stabilize, which means that there in an index $\lambda_0 < \omega_1$ such that $V_\lambda = V_{\lambda_0}$ for any $\lambda \geq \lambda_0$. By construction, this is only possible if $V_{\lambda_0} = X_{\alpha_0}$. Let $\beta = \alpha_{\lambda_0}$ and $i^\beta_\alpha = i^{\lambda_0}_{\alpha_0}$. Clearly $i^\beta_\alpha(X_\alpha) \cap p_\beta(F) = \emptyset$. □

We also need the following statement.

Lemma 2.2. *Let a non-metrizable compact space X be represented as the limit of an ω-spectrum $\mathcal{S} = \{X_\alpha, p^\beta_\alpha, A\}$ with soft projections p^β_α. Suppose that F is a closed subset of X containing no closed G_δ-subsets of X. Then for each $\alpha \in A$ there exists an index $\beta \in A$, with $\beta > \alpha$, such that $p_\beta(F)$ is a fibered Z-set in X_β with respect to the projection $p^\beta_\alpha\colon X_\beta \to X_\alpha$.*

Proof. Let $\alpha_0 = \alpha$. Choose $\alpha_{k+1} > \alpha_k$ so that the projection $p^{\alpha_{k+1}}_{\alpha_k} \colon X_{\alpha_{k+1}} \to X_{\alpha_k}$ has a section $i^{k+1}_k \colon X_{\alpha_k} \to X_{\alpha_{k+1}}$ such that $i^{k+1}_k(X_{\alpha_k}) \cap p_{\alpha_{k+1}}(F) = \emptyset$. Let $\beta = \sup\{\alpha_k \colon k \in \omega\}$. Let us show that $p_\beta(F)$ is a fibered Z-set in X_β with respect to the projection p^β_α. Let $\mathcal{U} = \{U_i \colon i \in I\}$ be an open cover of X_β. Without loss of generality we may assume that $U_i = (p^\beta_{\alpha_k})^{-1}(U^k_i)$, $i \in I$, where $k \in \omega$ and U^k_i is open in X_{α_k}. Let $j \colon X_{\alpha_{k+1}} \to X_\beta$ be any section of the projection $p^\beta_{\alpha_{k+1}} \colon X_\beta \to X_{\alpha_{k+1}}$. Consider the map $f_{\mathcal{U}} = j i^{k+1}_k p^\beta_{\alpha_k} \colon X_\beta \to X_\beta$. Since

$$p^\beta_{\alpha_k} f_{\mathcal{U}} = p^\beta_{\alpha_k} j i^{k+1}_k p^\beta_{\alpha_k} = p^{\alpha_{k+1}}_{\alpha_k}(p^\beta_{\alpha_{k+1}} j) i^{k+1}_k p^\beta_{\alpha_k} = (p^{\alpha_{k+1}}_{\alpha_k} i^{k+1}_k) p^\beta_{\alpha_k} = p^\beta_{\alpha_k},$$

it follows that $f_{\mathcal{U}}$ is $\mathcal{U}$-close to id_{X_β}. Also $p^\beta_\alpha f_{\mathcal{U}} = p^{\alpha_k}_\alpha p^\beta_{\alpha_k} f_{\mathcal{U}} = p^{\alpha_k}_\alpha p^\beta_{\alpha_k} = p^\beta_\alpha$ (i.e. $f_{\mathcal{U}}$ acts fiberwise with respect to p^β_α). It only remains to note $f_{\mathcal{U}}(X_\beta) \cap p_\beta(F) = \emptyset$. □

Lemma 2.3. *Let X be a pseudocompact space without G_δ-points. If βX - its Stone-Čech compactification - is an absolute retract of weight ω_1, then βX is homeomorphic to I^{ω_1}.*

Proof. By assumption, if $x \in X$, then x is not a G_δ-point in βX. Since X is pseudocompact, no point in $\beta X \setminus X$ is a G_δ-subset in βX (see [3, Exercise 6I.1]). Thus βX has no G_δ-points. By Ščepin's theorem (see [1, Theorem 7.2.9]), $\beta X \approx I^{\omega_1}$. □

3. Main result

In this section we prove our main result.

Theorem 3.1. *Let X be a retract of Σ. Then the following conditions are equivalent:*

(i) *X is homeomorphic to Σ,*
(ii) *X has no G_δ-points.*

Proof. The implication (i) ⇒ (ii) is trivial.

(ii) ⇒ (i). Let $|A| = \omega_1$. First let us introduce some notation. If $C \subset B \subseteq A$, then $\pi_B \colon (I^\omega)^A \to (I^\omega)^B$ and $\pi^B_C \colon (I^\omega)^B \to (I^\omega)^C$ denote the corresponding projections. Similarly by $\lambda_B \colon (I^\omega)^B \to (I^\omega)^A$ and $\lambda^B_C \colon (I^\omega)^C \to (I^\omega)^B$ we denote the sections of π_B and π^B_C defined as follows:

$$\lambda_B(\{x_t \colon t \in B\}) = (\{x_t \colon t \in B\}, \{0_t \colon t \in A \setminus B\})$$

and

$$\lambda^B_C(\{x_t \colon t \in C\}) = (\{x_t \colon t \in C\}, \{0_t \colon t \in B \setminus C\}).$$

Here 0_t denotes the point in the t-th copy of the Hilbert cube, all coordinates of which ere equal to 0. Note that $\Sigma = \bigcup\{\lambda_B((I^\omega)^B) \colon B \in \exp_\omega A\}$.

Let $X \subseteq \Sigma$ and $r\colon \Sigma \to X$ be a retraction. Recall that Σ is normal, pseudocompact and $\beta\Sigma = (I^\omega)^A$ (see [2, Problems 2.7.14, 3.10.E and 3.12.23(c)]). Consequently, $\beta X = \mathrm{cl}_{(I^\omega)^A} X$ and r has the extension $\tilde{r}\colon (I^\omega)^A \to \mathrm{cl}_{(I^\omega)^A} X = Y$. Note that $\tilde{r}$ is also a retraction and consequently Y is a compact absolute retract. Note that X, as a retract of Σ, is pseudocompact. Therefore, by Lemma 2.3, $Y \approx I^{\omega_1}$.

For each $C, B \subseteq A$, with $C \subseteq B$, let $Y_B = \pi_B(Y)$, $p_B = \pi_B|Y$ and $p^B_C = \pi^B_C|Y_B$. Clearly, Y is the limit space of the ω-spectrum $\mathcal{S}_Y = \{Y_B, p^B_C, \exp_\omega A\}$. Since $Y \approx I^{\omega_1}$, Ščepin's spectral theorem (see [1, Theorem 1.3.4]) for ω-spectra insures that there exists an ω-closed and cofinal subset $\mathcal{A} \subseteq \exp_\omega A$ such that

(1) $Y = \lim \mathcal{S}_{\mathcal{A}}$, where $\mathcal{S}_{\mathcal{A}} = \{Y_B, p^B_C, \mathcal{A}\}$,
(2) $Y_B \approx I^\omega$ whenever $B \in \mathcal{A}$,
(3) $p^B_C\colon Y_B \to Y_C$ is a trivial fibration with fiber I^ω, whenever $C \subseteq B$, $C, B \in \mathcal{A}$.

Applying the same spectral theorem to the map $\tilde{r}\colon (I^\omega)^A \to Y$ and to the ω-spectra $\mathcal{S} = \{(I^\omega)^B, \pi^B_C, \mathcal{A}\}$ (whose limit is $(I^\omega)^A$) and $\mathcal{S}_{\mathcal{A}}$ we can find an ω-closed and cofinal subset $\mathcal{B} \subseteq \exp_\omega A$ such that $\mathcal{B} \subseteq \mathcal{A}$ and for each $B \in \mathcal{B}$ there exists a retraction $r_B\colon (I^\omega)^B \to Y_B$ such that $p_B\tilde{r} = r_B\pi_B$.

Let $B \in \mathcal{B}$ and consider the composition $i_B = \tilde{r}\lambda_B|Y_B\colon Y_B \to Y$. Note that $p_B i_B = p_B\tilde{r}\lambda_B|Y_B = r_B\pi_B\lambda_B|Y_B = r_B|Y_B = \mathrm{id}_{Y_B}$. In other words, i_B is a section of the projection p_B. If $C \subseteq B$, $C, B \in \mathcal{B}$, we let $i^B_C = p_B i_C$. Note that i^B_C is a section of the projection p^B_C.

Next we show that $X = \bigcup\{i_B(Y_B)\colon B \in \mathcal{B}\}$. Indeed, let $x \in X$. Since $\Sigma = \bigcup\{\lambda_B((I^\omega)^B)\colon B \in \mathcal{B}\}$, there is $B \in \mathcal{B}$ such that $x = \lambda_B(y)$ for some $y \in (I^\omega)^B$. But $y = \pi_B(\lambda_B(y)) = \pi_B(x) \subseteq \pi_B(X) \subseteq \pi_B(Y) = Y_B$. Consequently, $i_B(y) = \tilde{r}(\lambda_B(y)) = \tilde{r}(x) = r(x) = x$.

Next we will construct a cofinal collection of countable subsets $\{A_\alpha\colon \alpha < \omega_1\} \subseteq \mathcal{B}$ of A and homeomorphisms $h_\alpha\colon Y_{A_\alpha} \to (I^\omega)^{A_\alpha}$ satisfying the following conditions:

(i) $A_\alpha \subseteq A_\beta$, whenever $\alpha < \beta < \omega_1$;
(ii) $A_\beta = \bigcup\{A_\alpha\colon \alpha < \beta\}$, whenever $\beta < \omega_1$ is a limit ordinal;
(iii) For each $\alpha < \omega_1$, $\pi^{A_{\alpha+1}}_{A_\alpha} h_{\alpha+1} = h_\alpha p^{A_{\alpha+1}}_{A_\alpha}$, i.e. the following diagram is commutative

$$\begin{tikzcd}
Y_{A_{\alpha+1}} \arrow[r, "h_{\alpha+1}"] \arrow[d, "p^{A_{\alpha+1}}_{A_\alpha}"'] & (I^\omega)^{A_{\alpha+1}} \arrow[d, "\pi^{A_{\alpha+1}}_{A_\alpha}"] \\
Y_{A_\alpha} \arrow[r, "h_\alpha"] & (I^\omega)^{A_\alpha}
\end{tikzcd}$$

(iv) $h_\beta = \lim\{h_\alpha\colon \alpha < \beta\}$, whenever $\beta < \omega_1$ is a limit ordinal;
(v) $p^{A_{\alpha+1}}_{A_\alpha}\colon Y_{A_{\alpha+1}} \to Y_{A_\alpha}$ is a trivial fibration with fiber I^ω;

(vi) $i_{A_\alpha}^{A_{\alpha+1}}(Y_{A_\alpha})$ is a fibered Z-set in $Y_{A_{\alpha+1}}$ with respect to the projection $p_{A_\alpha}^{A_{\alpha+1}}$;

(vii) For each $\alpha < \omega_1$, $h_{\alpha+1} i_{A_\alpha}^{A_{\alpha+1}} = \lambda_{A_\alpha}^{A_{\alpha+1}} h_\alpha$, i.e. the following diagram commutes:

$$\begin{array}{ccc} Y_{A_{\alpha+1}} & \xrightarrow{h_{\alpha+1}} & (I^\omega)^{A_{\alpha+1}} \\ {\scriptstyle i_{A_\alpha}^{A_{\alpha+1}}}\Big\uparrow & & \Big\uparrow{\scriptstyle \lambda_{A_\alpha}^{A_{\alpha+1}}} \\ Y_{A_\alpha} & \xrightarrow{h_\alpha} & (I^\omega)^{A_\alpha} \,. \end{array}$$

Let A_0 be any element of $\mathcal{B}$ and take any homeomorphism $h_0\colon Y_{A_0} \to (I^\omega)^{A_0}$.

Let $\beta < \omega_1$. Suppose that for each $\alpha < \beta$ we have already constructed a countable set $A_\alpha \in \mathcal{B}$ and a homeomorphism $h_\alpha\colon Y_{A_\alpha} \to (I^\omega)^{A_\alpha}$ satisfying the above conditions for appropriate indices. We proceed by constructing these objects for the ordinal β.

If $\beta = \sup\{\alpha\colon \alpha < \beta\}$, then set $A_\beta = \cup\{A_\alpha\colon \alpha < \beta\}$ and $h_\beta = \lim\{h_\alpha\colon \alpha < \beta\}$. Then, all required conditions are clearly satisfied.

Now consider the case $\beta = \alpha + 1$. Since $i_{A_\alpha}(Y_{A_\alpha})$ is a metrizable compactum in Y, it cannot contain closed G_δ-subsets of Y (which contain copies of the Tychonov cube I^{ω_1}). Consequently, we can find, based on Lemma 2.2, an element $A_{\alpha+1} \in \mathcal{B}$, such that $A_\alpha \subseteq A_{\alpha+1}$ and $i_{A_\alpha}^{A_{\alpha+1}}(Y_{A_\alpha}) = p_{A_{\alpha+1}}(i_{A_\alpha}(Y_{A_\alpha}))$ is a fibered Z-set with respect to the projection $p_{A_\alpha}^{A_{\alpha+1}}$. Since both projections $p_{A_\alpha}^{A_{\alpha+1}}$ and $\pi_{A_\alpha}^{A_{\alpha+1}}$ are trivial fibrations with fiber I^ω, there exists a homeomorphism $f\colon Y_{A_{\alpha+1}} \to (I^\omega)^{A_{\alpha+1}}$ such that $\pi_{A_\alpha}^{A_{\alpha+1}} f = h_{A_\alpha} p_{A_\alpha}^{A_{\alpha+1}}$. Then the set $f(i_{A_\alpha}^{A_{\alpha+1}}(Y_{A_\alpha}))$ is a fibered Z-set in $(I^\omega)^{A_{\alpha+1}}$ with respect to the projection $\pi_{A_\alpha}^{A_{\alpha+1}}$. Consider now another fibered Z-set in $(I^\omega)^{A_{\alpha+1}}$ (also with respect to the projection $\pi_{A_\alpha}^{A_{\alpha+1}}$) – namely, $\lambda_{A_\alpha}^{A_{\alpha+1}}((I^\omega)^{A_\alpha})$. There is a homeomorphism

$$g\colon f(i_{A_\alpha}^{A_{\alpha+1}}(Y_{A_\alpha})) \to \lambda_{A_\alpha}^{A_{\alpha+1}}((I^\omega)^{A_\alpha})$$

which acts fiberwise (i.e. $\pi_{A_\alpha}^{A_{\alpha+1}} g = \pi_{A_\alpha}^{A_{\alpha+1}}$). Here is the expression for g:

$$g = \lambda_{A_\alpha}^{A_{\alpha+1}} h_\alpha p_{A_\alpha}^{A_{\alpha+1}} f^{-1} | f(i_{A_\alpha}^{A_{\alpha+1}}(Y_{A_\alpha})).$$

By the fibered Z-set unknotting theorem [5], g extends to a homeomorphism $G\colon (I^\omega)^{A_{\alpha+1}} \to (I^\omega)^{A_{\alpha+1}}$ such that $\pi_{A_\alpha}^{A_{\alpha+1}} G = \pi_{A_\alpha}^{A_{\alpha+1}}$ (i.e. G acts fiberwise). Then the required homeomorphism $h_{A_{\alpha+1}}$ is defined as the composition $Gf\colon Y_{A_{\alpha+1}} \to (I^\omega)^{A_{\alpha+1}}$. Straightforward verification shows that all the needed properties are satisfied.

This completes the inductive process. Now let $h = \lim\{h_\alpha \colon \alpha < \omega_1\}$. It is easy to see that $h \colon Y \to I^A$ is a homeomorphim such that $h(i_{A_\alpha}(Y_{A_\alpha})) = \lambda_{A_\alpha}((I^\omega)^{A_\alpha})$ for each $\alpha < \omega_1$. Consequently, $h(X) = \Sigma$. □

Corollary 3.2. *Let X be a retract of Σ. Then $X \times \Sigma$ is homeomorphic to Σ.*

Proof. Note that $X \times \Sigma$ is a retract of $\Sigma \times \Sigma \approx \Sigma$ and has no G_δ-points. □

Corollary 3.3. *The following conditions are equivalent for a compact space X:*

(i) *$X \times \Sigma$ is homeomorphic to Σ,*
(ii) *X is a metrizable absolute retract.*

Proof. (i) $\Rightarrow$ (ii). Let $h \colon \Sigma \to X \times \Sigma$ be a homeomorphism and $\pi \colon X \times \Sigma \to X$ be the projection. Clearly $r = \pi h \colon \Sigma \to X$ is a retraction. Since I^{ω_1} is the Stone-Čech compactification of Σ (and since X is compact), r admits the extension $\tilde{r} \colon I^{\omega_1} \to X$. Therefore X, as a retract of I^{ω_1}, is an absolute retract. Note also that X is separable (as an image of I^{ω_1}). But separable compact subspaces of Σ are metrizable.

(ii) $\Rightarrow$ (i). Apply Corollary 3.2. □

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